ADVANCES IN MATHEMATICS **110**, 47–87 (1995)

On a Functional Equation of A. Hurwitz*

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0. INTRODUCTION

In the unpublished notebooks of Hurwitz, under the date of December 6, 1918, one finds the following question: Why is it not possible for a power series

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

to become its derived series

$$P'(x) = a_1 + 2a_2 x + \cdots$$

after analytic continuation along some closed path [11, Vol. 2, p. 752]? It is clear that unless $P(x) = Ce^x$, P must be a multivalued function and as it is easily seen, any entire solution of the difference–differential equation

$$G(\zeta + 1) = \frac{1}{2i\pi} e^{-2i\pi\zeta} G'(\zeta) \tag{0.1}$$

provides a solution $P(x) = G((1/2i\pi) \operatorname{Log} x)$ to Hurwitz's original question (cf. Section 1). In [13], Naftalevich showed by an iteration method, that for Q a polynomial of degree $n \geqslant 1$, the more general equation

$$H(\zeta + 1) = e^{Q(\zeta)} H'(\zeta) \tag{0.2}$$

* This research has been partly supported by the National Science Foundation.

0001-8708/95 $9.00

has, for each $\rho \geq n+1$ in the case $n>1$ and for $\rho>2$ in the case $n=1$, entire solutions of order ρ. The vector space of solutions is, therefore, of non-denumerable dimension. Furthermore, this equation has entire solutions of normal type with respect to the order $\rho=n+1$ and has no nontrivial solutions, either of order $\rho<n+1$ or of minimal type with respect to the order $\rho=n+1$.

In the same paper [13], the author mentioned that Lewy had obtained the explicit solution

$$P(x)=\int_0^{\infty} \exp[xt+(\log|t|-i\pi)^2/4i\pi]\,dt. \tag{0.3}$$

This function is analytic in the half plane $\operatorname{Re} x>0$ and can be continued analytically into other parts of the plane by shifting the path of integration. If the path is rotated in a counterclockwise direction by an angle of 2π radians, we find the derivative $P'(x)$. What caught our attention was the fact that there was no mention in [13] about the relation between the remarkable example of Lewy and the solutions obtained therein. In this paper, we show that the above function (0.3) generates in some sense, all the entire solutions of Eq. (0.1). In fact, there is an entire function G_0, closely related to (0.3), solution of (0.1) such that any other entire solution G can be written as

$$G(\zeta)=c\exp(e^{2i\pi\zeta})+\sum_{n\in\mathbb{Z}} c_n G_0(\zeta+n). \tag{0.4}$$

(see Section 5). Let us mention that, as far as we have been able to ascertain, the literature on functional equations in the complex plane, similar to (0.1), only deals with existence questions and not with explicit descriptions of all the entire solutions (cf., for instance, [3, 7].) On the other hand, an inspiration for us has been the work of de Bruijn [7] on the equation

$$H'(y)=e^{\alpha y+\beta}H(y-1) \tag{0.5}$$

considered for y real, $-\infty<c\leq y$, where $\alpha>0$ and $\beta\in\mathbb{C}$ are fixed parameters. This equation arises in analytic number theory. The change from $\alpha>0$ to $\alpha\in i\mathbb{R}^*$ introduces considerable difficulties, as will be seen below.

Finally, we would like to remark that in the present paper we have tried to make clear that the solutions of Hurwitz's problem are in some sense automorphic. In fact, the process of analytic continuation is just the action of the fundamental group of $\mathbb{C}\setminus\{0\}$, and it acts on those functions as a differential operator. It is then not surprising that when transforming Eq. (0.1) (resp., (0.2)) via the Mellin transformation (resp., Laplace) we find the functional equation of the Gamma function, and hence periodic functions appear naturally. In the case of the plane punctured at a finite number of points, which we are currently considering, one sees Fuchsian theta and zeta functions (cf. [15]). Our element G_0 corresponds to the basic solutions of Appell in his study of doubly-periodic functions [1, 2]. One can also see our decomposition of the Green function in Section 3 as being in the same spirit as the trace formula in [18]. There is also a clear link with the theory of resurgent functions [9]. On one hand, we give here a complete harmonic analysis of one of the simplest equations of resurgence. On the other hand, there is also a formal relationship. Ecalle's theory is intricately related to the method of Borel summability of series. If one tries to solve (0.2) by iteration one finds either convergent series [13] or series that are divergent but which converge after Borel summation. The series thus obtained are similar to the theta functions of Poincaré, and therefore, the representation of an arbitrary solution of Hurwitz' problem (for one or several punctures) in terms of these theta series appears as akin to the corresponding problem for automorphic functions (cf. [5, 15]).

We hope that this final remark will convey to the reader our impression that our solution to the problem of Hurwitz is not only of interest per se, but that it also suggests a completely new class of harmonic analysis problems.

1. Preliminaries

Let us first explain the relation between Eq. (0.1) and Hurwitz's question. Assume that the power series $g(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence exactly equal to 1, has the property that there is an analytic continuation along the circle $|z-1| = 1$ described in the positive sense, to a function $\tilde{g}$, holomorphic in a neighborhood of $z = 0$ and such that

$$\tilde{g}(z) = g'(z) \qquad \text{for} \quad |z| < \varepsilon. \tag{1.1}$$

Then it follows that the function $\tilde{g}$ can be continued along the same circle as many times as we want, as long as we continue in the positive direction. Hence, introducing the change of variables $z - 1 = e^{2i\pi\zeta}$, the function G defined in a small neighborhood of $\zeta = 1/2$ by

$$G(\zeta) = g(z), \qquad z = 1 + e^{2i\pi\zeta}, \tag{1.2}$$

is not only holomorphic in that neighborhood, but has an analytic continuation to the half strip

$$\operatorname{Re} \zeta > \tfrac{1}{2} - \delta, \qquad |\operatorname{Im} \zeta| < \delta,\ \delta > 0.$$

Furthermore, in this half strip, it satisfies the equation

$$G(\zeta+1)=\frac{1}{2i\pi}e^{-2i\pi\zeta}G'(\zeta). \tag{1.3}$$

Now, one can use this equation to analytically continue the function G to the left, i.e., to a holomorphic function in the whole strip $|\operatorname{Im}\zeta|<\delta$. This implies in return that the original function g can be analytically continued indefinitely along the circle $|z-1|=1$, described in the positive and negative senses.

For this reason, it is natural to consider Hurwitz's question only for the class of power series g that have analytic continuations along any path starting at the origin which avoid $z=1$. This corresponds to the study of the solutions to (1.3) which are entire functions of ζ. The trivial solutions $g(z)=Ce^{z}$ correspond to the periodic solutions $G(\zeta)=C\exp(e^{2i\pi\zeta})$. We shall study Eq. (1.3) in the next section, but for the time being, we would like to discuss the meaning of Hans Lewy's explicit solution.

There is no indication in [13] about how Lewy arrived at the formula (0.3). We think that it was inspired by his work on waves in shallow water (cf. [12, 14]). The reason is that there are two natural ways of attacking the problem posed by Hurwitz in the z-plane. One tries to find solutions that are Laplace type integrals or Mellin-type integrals. Let us explain briefly the first approach. To simplify the notation, we continue analytically a function g along circles centered at the origin (and not at $z=1$ as above) and denote its analytic continuation by $g(ze^{2i\pi})$. In order to find solutions to Hurwitz's problem, let us suppose that $g(z)$ can be represented in the form

$$g(z)=\int_{\gamma_z} e^{zt}\varphi(t)\,dt; \tag{1.4}$$

the path γ_z, to be determined later, depends on z and on the conditions imposed on $g(z)$ and the behavior of the (multi-valued) holomorphic function $\varphi(t)$, and must be chosen so that the integral converges. Under natural assumptions on the choice of γ_z, we have that the Hurwitz equation $g(ze^{2i\pi})=g'(z)$ becomes

$$\int_{\gamma_z} e^{zt}(t\varphi(t)-\varphi(te^{2i\pi}))\,dt=0. \tag{1.5}$$

This equation is satisfied if $\varphi(t)$ satisfies

$$t\varphi(t)=\varphi(te^{2i\pi}) \qquad (t\in\gamma_z). \tag{1.6}$$

Taking logarithms, we find that (1.6) is a jump relation for the function $\Phi(t) = \operatorname{Log} \varphi(t)$, namely,

$$\Phi(te^{2i\pi}) - \Phi(t) = \operatorname{Log} t \qquad (t \in \gamma_z). \tag{1.7}$$

Let us consider a concrete example. A suitable path γ_z is the real line $t \to te^{i \arg z}$, $t \in (0, \infty)$. To solve (1.7) for this path, we recall that for an integrable function w on the real line, the Cauchy integral W,

$$W(z) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} \frac{w(t)}{t-z}\, dt,$$

is holomorphic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies

$$W(t+i0) - W(t-i0) = w(t), \tag{1.8}$$

where $W(t \pm i0)$ denote the limits from above (resp. below). If $w(t) = 0$ for $t < 0$, the function W is actually holomorphic in $\mathbb{C}\backslash[0, \infty[$. Now, the function $l(t)$ given by

$$l(t) = \begin{cases} \log t, & t > 0, \\ 0, & t < 0, \end{cases}$$

is not integrable in $\mathbb{R}$, but we can take a function k, holomorphic in $\mathbb{C}\backslash]-\infty, 0]$ such that $w(t) = k(t)\, l(t)$ is integrable in $\mathbb{R}$ so that (1.8) becomes

$$W(t+i0) - W(t-i0) = k(t)\, l(t), \qquad t \in \mathbb{R}\backslash\{0\}.$$

If the function k is non-vanishing in $\mathbb{C}\backslash]-\infty, 0]$, the function $\phi(z) = W(z)/k(z)$ gives a solution of the jump relation (1.7) for any $t \in]0, \infty[$. A very simple function Φ can be found by means of the identity

$$\int_0^\infty \frac{\log t}{(t+a)(t+b)}\, dt = \frac{\operatorname{Log}^2 a - \operatorname{Log}^2 b}{2(b-a)},$$

which holds for any two complex numbers a, b with arguments in $(-\pi, \pi)$, and $\operatorname{Log} z$ represents the principal value of the logarithm. Let $b = 1$ and $a = -z$, $z \in \mathbb{C}\backslash[0, \infty[$, and choose $k(t) = 1/(1+t)$, then we have $\Phi(z) = (\operatorname{Log}(-z))^2/4i\pi$. The formula (1.4) becomes in this case

$$P_0(z) = \int_0^\infty e^{zt} \exp((\operatorname{Log} t - i\pi)^2/4i\pi)\, dt, \qquad (\operatorname{Re} z < 0), \tag{1.9}$$

which is precisely Hans Lewy's solution (0.3). Note that the derivatives $P_0^{(n)}$ of P_0 are just the solutions obtained by multiplying $\varphi(t) = \exp((\operatorname{Log} t - i\pi)^2/4i\pi)$ by t^n.

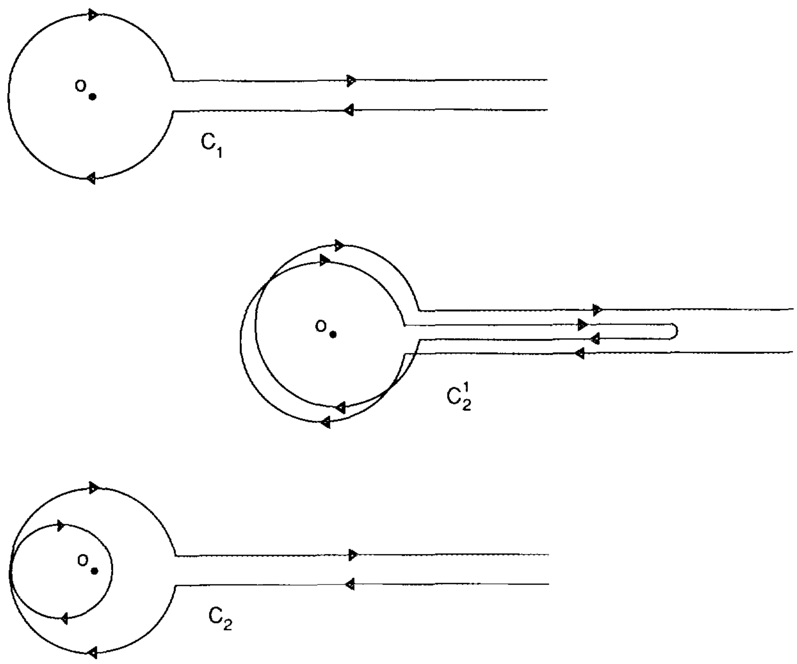

FIGURE 1.1

Another simply way of obtaining other solutions consists in taking the paths γ as (the k-fold of) a Hankel type contour, e.g., one of the contours in Fig. 1.1. The corresponding functions $P_k(z)$ are related to P_0 by the relations

$$P_k = \sum_{j=0}^{2k} (-1)^j P_0^{(j)}, \tag{1.10}$$

where $P_0^{(j)}$ is the jth derivative of P_0. (Note that the function φ has not changed.)

In the next section we look at the method of Mellin-type integrals to solve (1.3).

2. THE EQUATION $G(\zeta+1) = (1/2i\pi)\, e^{-2i\pi\zeta} G'(\zeta)$

Seeking particular solutions of Hurwitz's equation in the form of Mellin-type integrals is the same as looking for particular entire solutions of (0.1) of the form

$$G(\zeta) = \int_C e^{-2i\pi u\zeta} \varphi(u)\, du \tag{2.1}$$

for a convenient choice of holomorphic function φ and contour of integration C in the u-plane. If C' is the translate of C by 1 to the right, then by formal calculations,

$$\frac{1}{2i\pi} e^{-2i\pi\zeta} G'(\zeta) = -\int_C u\, e^{-2i\pi(u+1)\xi} \varphi(u)\, du$$

$$= -\int_{C'} (u-1)\, e^{-2i\pi\zeta u} \varphi(u-1)\, du,$$

and, if we could apply Cauchy's theorem, then we would have

$$\frac{1}{2i\pi} e^{-2i\pi\zeta} G'(\zeta) = -\int_C (u-1)\, e^{-2i\pi u\zeta} \varphi(u-1)\, du.$$

In this way, finding solutions of Eq. (0.1) is reduced to finding holomorphic solutions of the equation

$$e^{-2i\pi u} \varphi(u+1) = -u\varphi(u). \tag{2.2}$$

To eliminate the exponential factor, we write

$$\varphi(u) = e^{Au^2 + Bu} \Phi(u).$$

This leads to the choice $A = i\pi$, $B = 0$, and a function Φ which must satisfy

$$\Phi(u+1) = u\Phi(u).$$

Therefore, $\Phi(u) = \Gamma(u)\, \psi(u)$ where $\Gamma(u)$ is the classical Gamma function and $\psi(u)$ any periodic holomorphic function. The simplest choice is $\psi(u) \equiv 1$, hence we arrive at solution

$$G_0(\zeta) = \int_C e^{-2i\pi u\zeta + i\pi u^2} \Gamma(u)\, du \tag{2.3}$$

for a convenient choice of the contour C. To avoid the poles $0, -1, -2, \ldots$ of the meromorphic function Γ, we take as C a straight line through a point $z = a > 0$ of slope θ, $0 < \theta < \pi/2$. The well-known asymptotic representation

$$\Gamma(u) = \exp\left[\left(u - \frac{1}{2}\right) \operatorname{Log} u - u + \frac{1}{2} \log 2\pi\right]$$

$$\times \left(1 + O\left(\frac{1}{|u|}\right)\right), \qquad |\arg u| < \pi,$$

implies that the function G_0 is entire and that the previous reasoning is amply justified, thus G_0 is a solution of (0.1). Clearly G_0 is independent of the choice of $a > 0$, and $0 < \theta < \pi/2$. This function will be our fundamental building block and it is clear that its translates,

$$G_n(\zeta) = G_0(\zeta + n) \qquad (n \in \mathbb{Z}), \tag{2.4}$$

are also solutions of the same equation.

In order to study the asymptotic behavior of the function G_0, we need to find several different integral representations of it. This is given by the following.

LEMMA 1. *G_0 has the two additional representations:*

$$G_0(\zeta) = e^{i\pi/4} \int_{-\infty}^{+\infty} \exp\left[\frac{i}{4\pi}(2i\pi\zeta - w)^2 - e^w\right] dw \qquad (\operatorname{Re}\zeta > 0) \tag{2.5}$$

and

$$G_0(\zeta) = e^{i\pi/4} \int_{L_\theta} \exp\left[\frac{i}{4\pi}(2i\pi\zeta - w)^2 - e^w\right] dw \qquad (\zeta \in \mathbb{C}), \tag{2.6}$$

where for $0 < \theta < \pi/2$, L_θ is the contour shown in Fig. 2.1.

Proof. For the proof of this lemma, we take as a special contour C in (2.3) the straight line through the point $u = 1/2$ and of slope $\pi/4$. We use de Hankel's formula for the Gamma function [17, p. 245],

$$\Gamma(u) = \frac{-1}{2i \sin \pi u} \int_{+\infty}^{(0^+)} e^{-t}(-t)^{u-1}\, dt,$$

where the loop of integration consists of three parts: the t-plane has been cut along the positive real axis and t runs from $+\infty$ to $+\varepsilon$ on the upper border of the cut, then around the origin on a circle of radius ε in the positive sense and returns to $+\infty$ on the lower border of the cut.

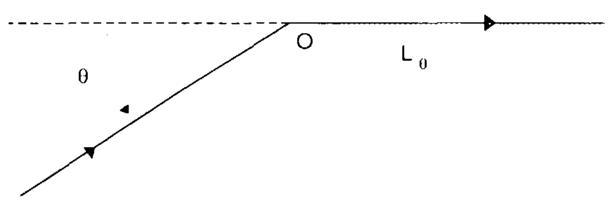

FIGURE 2.1

Therefore, after inverting the order of integration, which is allowed by the absolute convergence, we have

$$G_0(\zeta) = -\frac{1}{2i}\int_{+\infty}^{(0^+)} \frac{e^{-t}}{t}\left(\int_C \frac{e^{-2i\pi\zeta u + i\pi u^2}}{\sin \pi u}(-t)^u\, du\right) dt.$$

If we parameterize C by $u = 1/2 + e^{i\pi/4}v$, $v \in \mathbb{R}$ and denote by $\psi(\zeta, t)$ the integral

$$\psi(\zeta, t) = \int_{-\infty}^{+\infty} \exp[-\pi v^2 - 2i\pi\zeta v e^{i\pi/4} + i\pi v e^{i\pi/4}]\,\frac{(-t)^{ve^{i\pi/4}}}{\cos(\pi v e^{i\pi/4})}\, dv, \quad (2.7)$$

we get

$$G_0(\zeta) = -\frac{e^{-i\pi\zeta}}{2}\int_{+\infty}^{(0^+)} e^{-t}(-t)^{-1/2}\psi(\zeta, t)\, dt. \quad (2.8)$$

In order to derive (2.5) from this formula, we need to investigate the function $t \to \psi(\zeta, t)$ for fixed $\zeta > 0$. To do this, we write $\psi(\zeta, t) = \psi^1(\zeta, t) + \psi^2(\zeta, t)$, where

$$\psi^1(\zeta, t) = \int_0^{+\infty} \exp[-\pi v^2 - 2i\pi\zeta v e^{i\pi/4} + i\pi v e^{i\pi/4}]\,\frac{(-t)^{ve^{i\pi/4}}}{\cos(\pi v e^{i\pi/4})}\, dv.$$

We now set $t = \varepsilon e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, $0 < \varepsilon < 1$, and after simple calculations, we find that the integrand in $\psi^1(\zeta, t)$ is dominated in modulus by an expression of the form

$$\frac{1}{|\cos(\pi v e^{i\pi/4})|}\, e^{-\pi v^2 + a\zeta v},$$

where a is a positive constant independent of ε. This implies that

$$\lim_{|t| = \varepsilon \to 0} \psi^1(\zeta, t) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0}\int_{C_\varepsilon} e^{-t}(-t)^{-1/2}\psi^1(\zeta, t)\, dt = 0. \quad (2.9)$$

Here C_ε is the circle of center the origin and radius ε.

The behavior of the integral $\psi^2(\zeta, t)$ is slightly more complicated and, to obtain a result like (2.9), we follow another route. First, we obtain through the substitution $v \to -v$,

$$\psi^2(\zeta, t) = \int_0^{+\infty} \exp[-\pi v^2 + ve^{i\pi/4}(2i\pi\zeta - i\pi - \log(-t))] \cdot \frac{1}{\cos(\pi v e^{i\pi/4})}\, dv.$$

For $v > 0$, we can write

$$\frac{1}{\cos(\pi v e^{i\pi/4})} = 2e^{i\pi(\sqrt{2}/2)v - \pi(\sqrt{2}/2)v} \sum_{n \geq 0} (-1)^n e^{i\pi n\sqrt{2}v - \pi n\sqrt{2}v}.$$

This series converges uniformly, hence integration and summation can be interchanged. We obtain

$$\psi^2(\zeta, t) = 2 \sum_{n \geq 0} (-1)^n \int_0^{+\infty} e^{-\pi v^2 - \gamma_n v}\, dv,$$

where

$$-\gamma_n = e^{i\pi/4}(2i\pi\zeta - i\pi - \log(-t)) + \left(i\pi\frac{\sqrt{2}}{2} - \pi\frac{\sqrt{2}}{2}\right)(1 + 2n).$$

From [11, p. 1064], it follows that

$$\psi^2(\zeta, t) = 2 \sum_{n \geq 0} (-1)^n \sqrt{2\pi}\, e^{\gamma_n^2/8\pi} D_{-1}(\gamma_n/\sqrt{2\pi}). \tag{2.10}$$

The function D_{-1} is the parabolic cylinder function of degree -1 (one can use also the Hermite function of degree -1). Now, we note that for large n, γ_n belongs to the sector $|\arg z| < \pi/2$ where the function D_{-1} satisfies the asymptotic representation [16, p. 1065]

$$D_{-1}(z) = e^{-z^2/4}\frac{1}{z}\left(1 + O\left(\frac{1}{|z|^2}\right)\right),$$

or, for large n,

$$e^{\gamma_n^2/8\pi} D_{-1}\left(\frac{\gamma_n}{\sqrt{2\pi}}\right) = \frac{\sqrt{2\pi}}{\gamma_n}\left(1 + O\left(\frac{1}{n^2}\right)\right).$$

This proves, by the classical Abel lemma, that the series in (2.10) converges, in fact, uniformly in $\theta \in [0, 2\pi]$ and $\varepsilon \in]0, 1]$. This enables us to obtain an important estimate for $\psi^2(\zeta, t)$. Writing simply a for $2i\pi\zeta - i\pi - \log(-t)$, we have

$$\gamma_n = -e^{i\pi/4}(a + i\pi(1 + 2n)),$$

and the series (2.10) takes the form

$$\psi^2(\zeta, t) = -4\pi e^{-i\pi/4} \sum_{n \geq 0} \frac{(-1)^n}{a + i\pi(1 + 2n)}(1 + \varepsilon_n),$$

$\varepsilon_n = O(1/n^2)$. This shows that when $\varepsilon = |t|$ is sufficiently small, say $\varepsilon \leqslant \varepsilon_0$, there is a positive constant A, depending only in ζ, such that

$$|\psi^2(\zeta, t)| \leqslant A\,|\mathrm{Log}\,|t|\,|.$$

Consequently,

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon} e^{-t}(-t)^{-1/2}\psi^2(\zeta, t)\,dt = 0. \tag{2.9bis}$$

Now, using the two limits (2.9) and (2.9bis), we can further transform our function $G_0(\zeta)$:

$$\begin{aligned} G_0(\zeta) = -\frac{1}{2i}\Bigg[\int_{\infty}^{0} \frac{e^{-t}}{-t}\Bigg(\int_C \frac{e^{-2i\pi\zeta u + i\pi u^2}}{\sin \pi u}\, e^{i\pi u} t^u\,du \\ + \int_0^{\infty} \frac{e^{-t}}{-t}\int_C \frac{e^{-2i\pi\zeta u + i\pi u^2}}{\sin \pi u}\, e^{i\pi u} t^u\,du\Bigg)\,dt\Bigg] \\ = \int_0^{+\infty} \frac{e^{-t}}{-t}\Bigg(\int_C e^{-2i\pi\zeta u + i\pi u^2} t^u\,du\Bigg)\,dt. \end{aligned}$$

The inner integral can be explicitly computed:

$$\int_C e^{i\pi u^2 - 2i\pi\zeta u}\, t^u\,du = e^{i\pi/4}\, e^{(i/4\pi)(2i\pi\zeta - \mathrm{Log}\, t)^2}.$$

We get finally that for $\zeta > 0$,

$$G_0(\zeta) = e^{i\pi/4}\int_0^{\infty} \frac{e^{-t}}{t}\, e^{(i/4\pi)(2i\pi\zeta - \mathrm{Log}\, t)^2}\,dt,$$

which is (2.5) at least for $\zeta > 0$, but the formula holds for an arbitrary ζ in the half plane $\mathrm{Re}\,\zeta > 0$ by analytic continuation since both sides are holomorphic functions of ζ, when $\mathrm{Re}\,\zeta > 0$.

It is easy to see that the right-hand member of (2.6) is analytic in the whole ζ-plane and it is equal to the right-hand member of (2.5) when $\mathrm{Re}\,\zeta > 0$, by virtue of analytic continuation. We have thus (2.6) from each $\zeta \in \mathbb{C}$ and the lemma is proved. ∎

Remark. As a consequence of (2.5), we get

$$G_n(\zeta) = e^{3i\pi/4}\int_{-\infty - 2i\pi n}^{+\infty - 2i\pi n} \exp\left[\frac{i}{4\pi}(2i\pi\zeta - w)^2 - e^{w}\right]dw \tag{2.11}$$

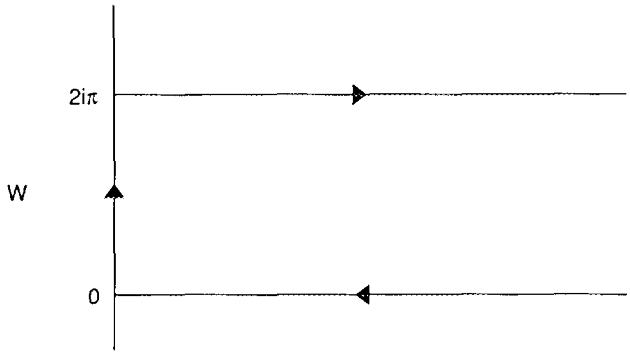

FIGURE 2.2

for $\zeta \in \mathbb{C}$, $\operatorname{Re} \zeta > -n$, $\pm n = 0, 1, 2, \ldots$. Moreover,

$$G_0(\zeta - 1) - G_0(\zeta)$$
$$= e^{3i\pi/4} \int_W \exp\left[\frac{i}{4\pi}(2i\pi\zeta - w)^2 - e^w\right] dw, \qquad \operatorname{Re}\zeta > 1 \quad (2.12)$$

The new path W is given in Fig. 2.2. The equality (2.12) is in fact true for every $\zeta \in \mathbb{C}$, by analytic continuation, since both sides of (2.12) are entire functions of ζ. It will be used to obtain the asymptotic behavior of $G_0(\zeta)$ in the complex plane.

3. THE ADJOINT EQUATION AND THE GREEN FUNCTION

We introduce in this section, following Borden (cf. [3, 6, 7]) the concept of the adjoint equation. Consider the linear difference–differential operator

$$LG(\zeta) = G'(\zeta) - 2i\pi e^{2i\pi\zeta} G(\zeta + 1), \qquad (3.1)$$

so that Eq. (1.3) is just $LG(\zeta) = 0$. We define the adjoint operator L^* by

$$L^*F(\zeta) = -F'(\zeta) - 2i\pi\, e^{2i\pi\zeta} F(\zeta - 1), \qquad (3.2)$$

so that $L^*F(\zeta) = 0$ means that

$$F(\zeta - 1) = \frac{i}{2\pi} e^{-2i\pi\zeta} F'(\zeta). \qquad (3.3)$$

This equation will be referred as the adjoint equation of (1.3). It is remarkable that this equation arises through the substitution $z = e^{2i\pi\zeta}$ from the "adjoint Hurwitz's problem" (cf. (1.1)),

$$\tilde{\tilde{g}}(z) = -g'(z),$$

where $\tilde{\tilde{g}}(z)$ is the analytic continuation of g in the negative direction. It is easy to verify that for each z_1 and $_2$ in the complex plane,

$$\int_{z_1}^{z_2} [F(\zeta)\, LG(\zeta) - G(\zeta)\, L^*F(\zeta)]\, d\zeta = \varphi(z_2) - \varphi(z_1),$$

where

$$\varphi(z) = F(z)\, G(z) - 2i\pi \int_{z-1}^{z} e^{2i\pi x} F(x)\, G(x+1)\, dx. \tag{3.4}$$

In particular, if $LG = 0$ and $L^*F = 0$, then (3.4) shows that φ is a constant function. We define the inner product $\{f, g\}$ by the formula

$$\{f, g\} = f(z)\, g(z) - 2i\pi \int_{z-1}^{z} e^{2i\pi x} f(x)\, g(x+1)\, dx. \tag{3.5}$$

Now, following [3, 7], we make the

DEFINITION. The Green function associated to the operator L is a function $G(\cdot, \zeta)$ defined for each real ζ as a solution of

$$\begin{cases} G'(x, \zeta) = 2i\pi\, e^{-2i\pi x} G(x+1, \zeta), & \text{for} \quad x < \zeta, \\ G(x, \zeta) = 0, & \text{for} \quad x > \zeta, \\ G(\zeta, \zeta) = 1. \end{cases} \tag{3.6}$$

The derivative in (3.6) is taken with respect to x.

As it is shown in [4], if we define the function $G^*(\eta, \cdot)$ for each real η by

$$\begin{cases} G^{*\prime}(\eta, y) = -2i\pi\, e^{-2i\pi y} G^*(\eta, y-1), & \text{for} \quad y > \eta, \\ G^*(\eta, y) = 0 & \text{for} \quad y < \eta, \\ G^*(\eta, \eta) = 1, \end{cases} \tag{3.7}$$

and the derivative in (3.7) is taken with respect to y, then the functions G and G^* are identical. These functions, as we will show, enable us to express any (entire) solution of (1.3) and (3.3) in terms of their restrictions on horizontal segments of length 1. For the moment, we study in some detail the function G^* and its Laplace transform. We have, from (3.7),

$$G^*(\eta, y) = 1 - 2i\pi \int_\eta^y e^{2i\pi t} G^*(\eta, t-1)\, dt,$$

and if we introduce the function (for a fixed η) $m(y) = |G^*(\eta, y)|$, then

$$\begin{cases} m(y) \leqslant 1 + 2\pi \int_\eta^y m(t-1)\, dt, & y > \eta, \\ m(y) = 0, & y < \eta, \\ m(\eta) = 1. \end{cases} \tag{3.8}$$

To describe the behavior of the solutions of this functional inequality we introduce the exact solution m_0 of (3.8). It verifies

$$\begin{cases} m_0(y) = 1 + 2\pi \int_\eta^y m_0(t-1)\, dt, & y > \eta, \\ m_0(y) = 0, & y < \eta, \\ m_0(\eta) = 1. \end{cases}$$

We claim that for each real y, $m(y) \leqslant m_0(y)$. This is clear for $y \leqslant \eta$, and if $\eta < y < \eta + 1$, then

$$m(y) - m_0(y) \leqslant 2\pi \int_\eta^y (m(t-1) - m_0(t-1))\, dt \leqslant 0;$$

this implies the inequality $m(y) \leqslant m_0(y)$ for $y < \eta + 1$. By iteration, we have the inequality for $y < \eta + n$, for each integer n.

Let us now evaluate explicitly the function m_0. For a fixed η, define a sequence of functions $(f_n)_{n \geqslant 0}$ by

$$f_0(y) = 0, \qquad y < \eta,$$
$$f_0(y) = 1, \qquad y \geqslant \eta,$$

and for $n \geqslant 1$,

$$f_n(y) = \int_{-\infty}^{y-1} f_{n-1}(t)\, dt.$$

It is easily verified that for $y \geqslant \eta + n$,

$$m_0(y) = \sum_{n \leqslant y - \eta} (2\pi)^n f_n(y),$$

and by simple induction, we see that f_n is given by

$$\begin{cases} f_n(y)=0, & y<\eta+n, \\ f_n(y)=\dfrac{(y-\eta-n)^n}{n!}, & y\geqslant \eta+n. \end{cases}$$

We therefore have

$$\begin{cases} m_0(y)=\displaystyle\sum_{0\leqslant n\leqslant y-\eta} \frac{(2\pi)^n}{n!}(y-\eta-n)^n, & y\geqslant \eta, \\ m_0(y)=0, & y<\eta. \end{cases}$$

The term corresponding to $n=0$ has to be interpreted as 1 for $y=\eta$. Finally, from the formulas

$$m_0(y)=m_0(n)+\int_n^y m_0'(t)\,dt=m_0(n)+2\pi\int_{n-1}^{y-1} m_0(t)\,dt,$$

we obtain that for each integer n

$$\sup_{n\leqslant t\leqslant n+1} m_0(t)\leqslant (1+2\pi)\sup_{n-1\leqslant t\leqslant n} m_0(t).$$

Hence, for some positive constant $c(\eta)$ we have

$$m_0(x)\leqslant c(\eta)(1+2\pi)^x, \qquad x>0.$$

This means that for each fixed η, the function $G^*(\eta,\cdot)$ is of exponential type on the real axis. Therefore, for large t we get from (3.7) that

$$\int_\eta^\infty e^{-yt}G^{*\prime}(\eta,y)\,dy=-2i\pi\int_\eta^\infty e^{-yt}e^{2i\pi y}G^*(\eta,y-1)\,dy. \tag{3.9}$$

At this point, it is convenient to introduce the Laplace transform of $G^*(\eta,\cdot)$ defined by

$$F(t)=\int_\eta^\infty e^{-yt}G^*(\eta,y)\,dy.$$

Then Eq. (3.9) and integration by parts lead to a difference equation for F:

$$-2i\pi e^{-t}F(t-2i\pi)=tF(t)-e^{-\eta t}. \tag{3.10}$$

Formally, F admits the following expansion:

$$F(t)=e^{-\eta t}\sum_{n\geqslant 0}(-2i\pi)^n\frac{e^{2i\pi n\eta}\,e^{-nt}}{t(t-2i\pi)\cdots(t-2i\pi n)}. \tag{3.11}$$

This is a kind of Dirichlet series with rational functions as coefficients. The claim is that (3.11) is in fact the Laplace transform of G^*. For this purpose, we will have to take its inverse Laplace transform, so we need to know more about the analytic properties of (3.11). This is given by the following.

PROPOSITION 3.1. *For each $\delta_0 > 0$, the series (3.11) is absolutely and uniformly convergent in each half plane $\{t \in \mathbb{C},\ \operatorname{Re} t \geqslant \delta_0 > 0\}$. If $\delta_0 < 0$, for each $\alpha_0 > 0$, the series converges absolutely and uniformly in the set $\{t \in \mathbb{C},\ \operatorname{Re} t \geqslant \delta_0,\ |t - 2i\pi n| \geqslant \alpha_0,\ n = 0, 1, 2, \ldots\}$.*

Remark 3.2. The series $F(t)$ given by (3.11) has the remarkable property that it has isolated points of divergence.

Proof of Proposition 3.1. For an arbitrary integer n, we denote by $D_n(t)$ the polynomial

$$D_n(t) = t(t - 2i\pi) \cdots (t - 2in\pi).$$

To prove the proposition, it is enough to prove the second statement. Let $\delta_0 < 0$ and $B_k = B(2i\pi k, \alpha_0)$ the disk centered at $2i\pi k$ and of radius $0 < \alpha_0 < 1$. It is clear that for each $t \in \mathbb{C}$ with $\operatorname{Im} t \leqslant 0$ the inequality

$$\alpha_0 (2\pi)^n n! \leqslant |D_n(t)|$$

holds. If $\operatorname{Im} t > 0$, we introduce the non-negative integer k_0 such that $2\pi k_0 \leqslant \operatorname{Im} t < 2\pi(k_0 + 1)$ and we investigate $|D_n(t)|$ in the three cases $0 \leqslant n < k_0$, $n = k_0$, and $n > k_0$.

As can be easily seen from Fig. 3.1, we have: if $0 \leqslant n < k_0$,

$$(2\pi)^{n+1} k_0 (k_0 - 1) \cdots (k_0 - n) \leqslant |D_n(t)|,$$

hence

$$(2\pi)^{n+1} (n+1)! \leqslant |D_n(t)|.$$

If $n = k_0$ and $t \notin B_{k_0}$, then

$$\alpha_0 (2\pi)^n n! \leqslant |D_n(t)|.$$

Finally, if $n > k_0$ and $t \notin B_{k_0}$, then

$$\alpha_0^2 (2\pi)^{n-1} k_0!\, (n - k_0 - 1)! \leqslant |D_n(t)|.$$

Therefore

$$\alpha_0^2 (2\pi)^{n-1} \frac{(n-1)!}{2^n} \leqslant |D_n(t)|.$$

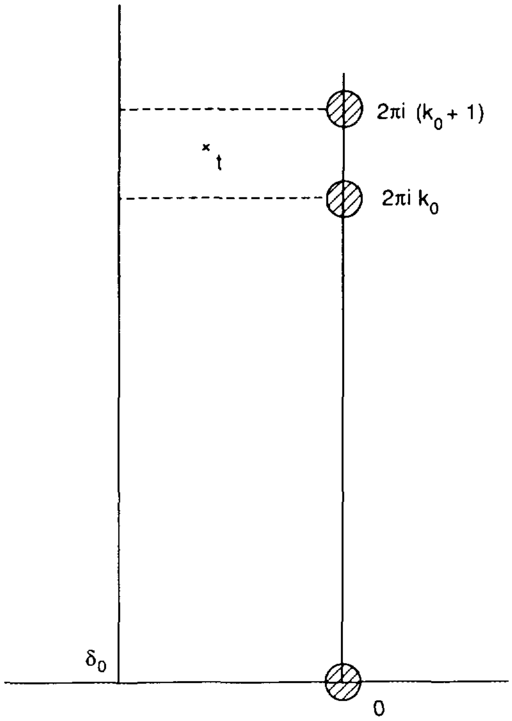

FIGURE 3.1

Hence, if $n \geqslant 1$, $0 < \alpha_0 < 1$ and $t \notin \bigcup_{k \geqslant 0} B_k$, we have the following inequality:

$$\alpha_0^2 (2\pi)^{n-1} \frac{(n-1)!}{2^n} \leqslant |D_n(t)|.$$

If, in addition, $\operatorname{Re} t \geqslant \delta_0$, we let $\delta = |\delta_0| + 2\pi\, |\eta|$ to obtain

$$\left| (-2i\pi)^n \frac{e^{2i\pi n\eta} \cdot e^{-nt}}{t(t-2i\pi)\cdots(t-2in\pi)} \right| \leqslant \frac{2\pi}{\alpha_0^2} \frac{e^{-n\delta} 2^n}{(n-1)!}.$$

The proposition is proved. ∎

We can now show that the inverse Laplace transform $H(y)$ of the series $F(t)$ is, in fact, $G^*(\eta, y)$ (η is fixed). First we observe that from the proof of Proposition 3.1, $|F(t)| = O(e^{-\eta t})$ when $t \to +\infty$. Since for $\gamma > 0$

$$H(y) = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{ty} F(t)\, dt,$$

it follows that supp $H \subseteq [\eta, \infty)$ (just apply Cauchy's theorem). Moreover, by Cauchy's theorem we have for any non-negative integer n

$$\frac{1}{2i\pi}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{(y-n-\eta)t}}{t(t-2i\pi)\cdots(t-2i\pi n)}\,dt$$

$$= \begin{cases} \dfrac{(e^{2i\pi(y-\eta)}-1)^n}{(2i\pi)^n n!}, & \text{if } 0 \leqslant n \leqslant y-\eta, \\ 0, & \text{if } n > y-\eta, \end{cases}$$

where the identity for $n=0$ has to be interpreted as equal to 1 when $y=\eta$. It follows from the formula (3.11) and Proposition 3.1 that $H(\eta)=1$ and

$$H(y) = \sum_{0 \leqslant n \leqslant y-\eta} \frac{(-1)^n}{n!} e^{2i\pi\eta n}(e^{2i\pi(y-\eta)}-1)^n$$

if $\eta < y$. It is now evident that H satisfies (3.7). By the uniqueness of the Green function and of the Laplace transform, we conclude that $H(y) = G^*(\eta, y)$ and that the series (3.11) is indeed the Laplace transform of $G^*(\eta, y)$. Replacing η by x, and recalling $G = G^*$, we summarize these observations by the formula

$$G(x, y) = \sum_{0 \leqslant n \leqslant y-x} \frac{(-1)^n}{n!} e^{2i\pi n x}(e^{2i\pi(y-x)}-1)^n, \qquad x < y, \qquad (3.12)$$

$G(x, x) = 1$, and $G(x, y) = 0$ if $x > y$.

We will need below another representation of F obtained with the help of the incomplete Gamma function [10, 16]. This function is defined by

$$\begin{aligned} \gamma(s, x) &= \int_0^x t^{s-1} e^{-t}\,dt \qquad (\operatorname{Re} s > 0, |\arg s| < \pi) \\ &= \int_0^x (x-t)^{s-1} e^{-(x-t)}\,dt \\ &= e^{-x} \sum_{k=0}^{\infty} \int_0^1 (x-xu)^{s-1} \frac{(xu)^k}{k!}\,d(xu) \\ &= e^{-x} \sum_{k=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s+k+1)} x^{s+k}. \end{aligned}$$

From this, we easily derive the formula

$$\gamma(t, e^{2i\pi(\eta+t)}) = -2i\pi \exp(-e^{2i\pi(\eta+t)})\, e^{2i\pi t^2} F(-2i\pi t),$$

which implies

$$F(t) = -\frac{1}{2i\pi} e^{-t^2/2i\pi} \exp(e^{2i\pi\eta - t}) \gamma\left(-\frac{t}{2i\pi}, e^{2i\pi\eta} e^{-t}\right). \tag{3.13}$$

Using the expansion [10, Vol. II, p. 135]

$$\gamma(s, x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{s+k}}{s+k},$$

we get another formula for $F(t)$:

$$F(t) = -\exp(e^{2i\pi\eta - t}) e^{-t\eta} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{e^{-nt} e^{2i\pi n\eta}}{2i\pi n - t}. \tag{3.14}$$

This representation furnishes another proof that the function $F(t)$ is meromorphic with simple poles at $t = 2i\pi n$, $n = 0, 1, 2, \ldots$.

Remark 3.3. Asymptotic representations of $F(t)$ can be obtained from (3.13) and the asymptotic representations of the incomplete Gamma function.

A more interesting expression than (3.12) for the Green function can be given in terms of the solutions G_n (cf. (2.4)) and some special solutions F_n of the adjoint equation (3.2). If we look for Laplace-type integrals solutions of (3.3), we find that the function

$$F_0(\zeta) = \int_C e^{a\zeta z + Az^2} \frac{dz}{\Gamma(1+z)}$$

verifies Eq. (3.3),

$$F_0(\zeta - 1) = \frac{i}{2\pi} e^{-2i\pi\zeta} F_0(\zeta),$$

provided that $a = 2i\pi$, $A = -i\pi$, and C is any path ending at infinity in the sectors

$$\left|\theta - \varepsilon\frac{\pi}{4}\right| \leqslant \frac{\pi}{4} - \delta, \qquad \varepsilon = -1, 3, 0 < \delta < \frac{\pi}{4}.$$

As in (2.4), the functions

$$F_n(\zeta) = F_0(\zeta + n), \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{3.15}$$

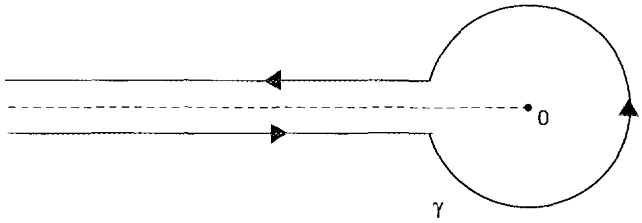

FIGURE 3.2

are also solutions of (3.3) and, as was the case for G_0, the function F_0 admits other integral representations. Let us recall the following familiar Hankel representation for the inverse Gamma function:

$$\frac{1}{\Gamma(1+z)}=\frac{1}{2i\pi}\int_{\gamma} e^{t}t^{-z-1}\,dt.$$

Here, γ is the loop represented in Fig. 3.2.

After substitution and interchanging the order of integration, we get the integral representation

$$F_0(\zeta)=\frac{1}{2i\pi}\int_{\gamma} e^{t}\frac{dt}{t}\int_{C} e^{2i\pi z\zeta-i\pi z^2-z\log t}\,dz.$$

The inner integral is a Gaussian integral, thus

$$F_0(\zeta)=\frac{1}{2i\pi}e^{-i\pi/4}\int_{\gamma} e^{t-(i/4\pi)(2i\pi\zeta-\log t)^2}\frac{dt}{t}. \tag{3.16}$$

Making the substitution $t=e^{w}$, we obtain

$$F_0(\zeta)=\frac{1}{2i\pi}e^{-i\pi/4}\int_{V} e^{-(i\pi/4)(2i\pi\zeta-w)^2+e^{w}}\,dw, \tag{3.17}$$

where V is the path indicated in the Fig. 3.3.

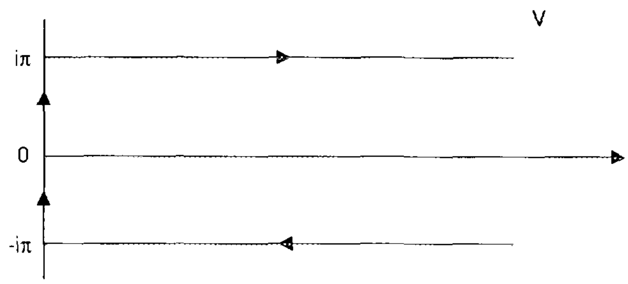

FIGURE 3.3

Remark 3.4. The representation (3.17) has some similarity with formula (2.12). It will be used to obtain the asymptotic expansion of $F_0(\zeta)$ for $|\arg \zeta| < \pi$.

Remark 3.5. There are periodic solutions of the equations (1.3) and (3.3). In fact, Eq. (1.3) has the solution

$$\tilde{G}(\zeta) = \exp(e^{2i\pi\zeta})$$

and Eq. (3.3) has the solution

$$\tilde{F}(\zeta) = \exp(-e^{2i\pi\zeta}).$$

Now we are ready to discuss how the Green function can be expressed in terms of the functions G_n, $\tilde{G}$ and F_n, $\tilde{F}$, $n = 0, \pm 1, \pm 2, \ldots$. This is given by the following theorem.

THEOREM 3.6. *Let y be a real number. Then we have for all $x > y - 1$ (except for $x = y$)*

$$2i\pi G(y, x) = 2i\pi \tilde{F}(x)\, \tilde{G}(y) - \sum_{n=-\infty}^{+\infty} F_n(x)\, G_n(y).$$

Proof. The proof may be obtained from (3.13) or (3.14). We prefer to use the integral representations (2.5) and (3.17). We first express $F_0(x)\, G_0(y)$ in the form of a single integral (as in [7]), with x and y complex numbers such that $x - y = t$ is real. Assume Re $y > 0$. Then from (2.5) we get

$$G_0(y) = e^{i\pi/4} \int_{-\infty}^{+\infty} e^{(i/4\pi)(2i\pi y - w)^2 - e^w}\, dw,$$

and, if w and η are real, then by Cauchy's theorem and (3.16),

$$F_0(x) = \frac{e^{-i\pi/4}}{2i\pi} \int_{V-\eta} e^{-(i/4\pi)(2i\pi x - z - w) + e^{z+w}}\, dz,$$

so that

$$F_0(x)\, G_0(y) = \frac{1}{2i\pi} \int_{-\infty}^{+\infty} dw \int_{V-\eta} e^{\phi(w,z)}\, dz, \tag{3.18}$$

where

$$\phi(w, z) = \frac{(z - 2i\pi t)^2}{4i\pi} - \frac{(z - 2i\pi t)(2i\pi y - w)}{2i\pi} + e^w(e^z - 1).$$

But for $z \in V - \eta$ and $|z|$ large, $\mathrm{Re}(e^z - 1)$ is equal to $-e^{\mathrm{Re}\, z} - 1$ (η is real) and the integral (3.18) is absolutely convergent if

$$\mathrm{Re}\left(\frac{z - 2i\pi t}{2i\pi}\right)$$

is positive for every z in $V - \eta$. This means that

$$\mathrm{Re}\left(\frac{x}{2i\pi} \pm \frac{1}{2} - t\right)$$

is positive for each real x or $t > -1/2$. In this case, we can change the order of integration and by using Euler's formula for the Gamma function, we find that

$$F_0(y+t)\, G_0(y) = \frac{1}{2i\pi} \int_{V-\eta} \exp\left\{y(2i\pi t - z) + \frac{(z - 2i\pi t)^2}{4i\pi}\right\}$$
$$\times (1 - e^z)^{(2i\pi t - z)/2i\pi}\, \Gamma\left(\frac{z - 2i\pi t}{2i\pi}\right) dz.$$

Moreover, if we assume $-1 < t$, then the function

$$f(z) = \exp\left\{y(2i\pi t - z) + \frac{(z - 2i\pi t)^2}{4i\pi}\right\}$$
$$\times (1 - e^z)^{(2i\pi t - z)/2i\pi}\, \Gamma\left(\frac{z - 2i\pi t}{2i\pi}\right) \tag{3.19}$$

has a convergent integral on any path leading to the origin. We can transform the path $V - \eta$ into a path coming from $+\infty$ to the origin and going back from the origin to $+\infty$. We thus have

$$F_0(y+t)\, G_0(y) = -\frac{1}{2i\pi} \int_0^{\infty} (f(ze^{2i\pi}) - f(z))\, dz$$
$$= -\frac{1}{2i\pi} \int_0^{\infty} \exp\left\{y(2i\pi t - z) + \frac{(z - 2i\pi t)^2}{4i\pi}\right\}$$
$$\times (e^z - 1)^{(2i\pi t - z)/2i\pi} \cdot 2i \sin \pi\left(t - \frac{z}{2i\pi}\right) \Gamma\left(\frac{z - 2i\pi t}{2i\pi}\right) dz,$$

which is, according to a well known formula for the Gamma function

$$= -\int_0^{\infty} \exp\left\{ y(2i\pi t - z) + \frac{(z-2i\pi t)^2}{4i\pi} \right\}$$

$$\times (e^z - 1)^{(2i\pi t - z)/2i\pi} \left(1 \Big/ \Gamma\left(1 + \frac{2i\pi t - z}{2i\pi}\right)\right) dz$$

$$= -2\pi \int_0^{\infty} \exp\{2i\pi y(t+is) + i\pi(t+is)^2\}(e^{2\pi s} - 1)^{t+is} \frac{ds}{\Gamma(1+t+is)}$$

or

$$F_0(y+t)\, G_0(y)$$

$$= 2i\pi \int_L \exp(2i\pi yp + i\pi p^2) \cdot (e^{2i\pi t - 2i\pi p} - 1)^p \frac{dp}{\Gamma(1+p)}, \qquad (3.20)$$

where L is the vertical half-line $[t+i0,\ t+i\infty[$. Now, in the last integral, we are free to move the path L to the horizontal half-line $[t, +\infty[$ since in the sector $0 < \arg p < \pi/2$, the Cauchy theorem applies. We then have

$$F_0(y+t)\, G_0(y) = 2i\pi \int_t^{+\infty} \tilde{\varphi}(x)\, dx, \qquad (3.21)$$

where $\tilde{\varphi}(x)$ denotes the integrand in (3.20). Using

$$F_n(x) = F_0(x+n), \qquad G_n(y) = G_0(y+n),$$

we obtain from (3.21)

$$F_n(y+t)\, G_n(y) = 2i\pi \int_t^{+\infty} e^{2i\pi xn} \tilde{\varphi}(x)\, dx. \qquad (3.22)$$

Now the left side of (3.20) is an entire function of t and the left side is an analytic function of t in the half-plane $\operatorname{Re} t > -1$. So, by analytic continuation, (3.20) and hence (3.21) and (3.22), remain valid for all t in the complex half-plane $\operatorname{Re} t > -1$, in particular, for all $t > -1$. We introduce the function φ defined by

$$\begin{cases} \varphi(x) = \tilde{\varphi}(x), & x > t \\ \varphi(x) = 0, & x \leqslant t. \end{cases} \qquad (3.23)$$

Then (3.22) takes the form

$$F_n(y+t)\, G_n(y) = 2i\pi \int_{-\infty}^{+\infty} e^{2i\pi nx} \varphi(x)\, dx.$$

The function $\tilde{\varphi}(x)$ is twice differentiable for $x \neq t$. If $t > 0$, $\tilde{\varphi}(x)$ is continuous at $x = t$ and if $-1 < t < 0$, it tends to infinity when x tends to t from the right, but the integral is still absolutely convergent. If $t = 0$, the function $\tilde{\varphi}(x)$ admits zero as limit when x tends to zero from the right and tends to infinity when x tends to zero from the left and all the other negative integers $k = -1, -2, ...$ are singularities of $\tilde{\varphi}(x)$. So, for $-1 < t$, $t \neq 0$, we can apply the Poisson summation formula to obtain

$$2i\pi \sum_{n=-\infty}^{+\infty} \varphi(n) = \sum_{n=-\infty}^{+\infty} F_n(y+t)\, G_n(y).$$

From (3.12) and (3.23), we conclude that

$$\sum_{n=-\infty}^{+\infty} F_n(y+t)\, G_n(y) = 2i\pi \sum_{t<n} \frac{(-1)^n}{n!} e^{2i\pi yn}(e^{2i\pi t} - 1)^n,$$

or

$$\sum_{n=-\infty}^{+\infty} F_n(y+t)\, G_n(y)$$
$$= 2i\pi \left\{ \exp(e^{2i\pi y} - e^{2i\pi x}) - \sum_{0 \leqslant n < t} \frac{(-1)^n}{n!} (e^{2i\pi x} - e^{2i\pi y})^n \right\} \tag{3.24}$$

for all $x > y - 1$, $x \neq y$, thus ending the proof of the theorem. ▮

4. Asymptotic Expansions

In this section, we want to study in some detail the behavior of the special solutions $G_n(x)$ and $F_n(x)$ for large values of $|n|$ when x is a fixed complex number. We also want to study the asymptotic behavior of these solutions when $|x|$ approaches infinity. By (2.4) and (3.15), it is sufficient to know the behavior of $G_0(x)$ and $F_0(x)$ in every region of the plane. We will solve these problems by means of saddle point analysis, but more preliminary work is necessary to deal with the function G_0. Before we accomplish this program, we now investigate the integral

$$F_\alpha(x) = \int_0^{+\infty} e^{-u} u^x e^{\alpha(\log u)^2}\, du, \tag{4.1}$$

where α is a complex number. This integral converges for $\operatorname{Re} x > -1$ and satisfies the functional equation

$$\begin{aligned} F_\alpha(x+1) &= (x+1)\, F_\alpha(x) + 2\alpha \frac{dF_\alpha}{dx}(x) \\ F_0(x) &= \Gamma(x+1). \end{aligned} \tag{4.2}$$

The function F_α is entire for $\alpha \neq 0$, $\pi/2 \leqslant \arg \alpha \leqslant 3\pi/2$. To see this, it is enough to prove that the integral

$$f_\alpha(x) = \int_0^1 e^{-u} u^x e^{\alpha(\log u)^2}\, du, \qquad \operatorname{Re} x > -1,$$

can be analytically continued to the whole complex plane. By means of the substitution $u = e^{-v}$, we find

$$f_\alpha(x) = \int_0^\infty \exp(e^{-v})\, e^{-v(x+1)+\alpha v^2}\, dv \qquad (\operatorname{Re} x > -1). \tag{4.3}$$

Now, in (4.3) we can rotate the contour of integration onto the ray $\arg \theta = (\pi/4) - (1/2) \arg \alpha$, we thus have

$$f_\alpha(x) = \int_0^1 \exp(-e^{-t \arg \alpha})\, e^{-t(x+1)e^{i \arg \alpha} - |\alpha| t^2}\, dt,$$

which represents an entire function of x. Furthermore, we note that the function (4.1) is also analytic in α, $\operatorname{Re} \alpha < 0$ and satisfies the heat equation

$$\begin{cases} \dfrac{\partial F_\alpha}{\partial \alpha}(x) = \dfrac{\partial^2 F_\alpha}{\partial x^2}(x), \\ F_0(x) = \Gamma(x+1). \end{cases} \tag{4.4}$$

To determine the asymptotic behavior of F_α in the sector $|\arg x| < \delta$, we use the following lemma (see, for example, [4, Satz 4]).

Lemma 4.1. *Let $\chi(t)$ be a continuous function for $t > 0$ and let $\chi(t)$ be twice differentiable for $t > t_1 > 0$. Let, moreover,*

$$\begin{gathered} \chi^{(\nu)}(t) = O(t^{\sigma-\nu}), \qquad t \to +\infty,\ \nu = 1, 2, \\ \int_0^{t_1} e^{\chi(t)}\, t^{\delta_0}\, dt < \infty, \end{gathered} \tag{4.5}$$

where $\alpha > 0$ and δ_0 is a certain fixed parameter. Then, for any $c > 0$,

$$\int_0^\infty e^{-ct^\sigma + \chi(t)}\, t^x\, dt \sim \sqrt{\frac{2\pi}{cs^\sigma}}\, \frac{s}{\sigma}\, e^{-cs+\chi(s)}\, s^x \qquad (x \to +\infty), \tag{4.6}$$

with s a solution of $x = c\sigma s^\sigma - s\chi'(s)$, i.e., a critical point of the phase function.

We take $c = \sigma = 1$ and $\chi(t) = \alpha(\log t)^2$ to get the integral (4.1). Hence

$$\int_0^{\infty} e^{-t + \alpha(\log t)^2} t^x \, dt \sim \sqrt{\frac{2\pi}{s}} s \, e^{-s + \alpha(\log s)^2} s^x \qquad (x \to +\infty)$$

with s a solution of

$$x = s - s\chi'(s) = s - 2\alpha \log s. \tag{4.7}$$

Note that $|s|$ tends to infinity when x tends to infinity because $\operatorname{Re} \alpha \leqslant 0$. It is easy to see that

$$s = x + 2\alpha \log x + \frac{(2\alpha)^2}{x} \log x + \frac{O(\log x)}{x^2}.$$

Hence, for $x \to +\infty$,

$$\int_0^{\infty} e^{-t + \alpha(\log t)^2} t^x \, dt$$
$$= \sqrt{2\pi} \exp\left\{\frac{1}{2} \log x - x + \alpha(\log x)^2 + x \log x\right\}$$
$$\times \left(1 + O\left(\frac{(\log x)^2}{x}\right)\right). \tag{4.8}$$

Now, to see that (4.8) remains valid for $|\arg x| < \delta < \pi/2$, we follow the usual techniques. We make the substitution $t = xu$ and rotate the path of integration.

Remark 4.2. The asymptotic relation (4.8) and the functional equation (4.2) can be used to give the complete asymptotic representation of (4.1).

The following two lemmas will be useful.

LEMMA 4.3. *Let ε be a small positive number, for any determination $\log_\alpha z = \operatorname{Log} |z| + i\theta$, $\alpha < \theta < \alpha + 2\pi$, the equation*

$$2i\pi e^{w} + w - 2i\pi x = 0 \tag{4.9}$$

has for $\varepsilon + \alpha < \arg x < 2\pi + \alpha - \varepsilon$ and $|x|$ large, a solution $\zeta = \zeta_\alpha$ given by

$$\zeta = \log_\alpha x - \frac{\log_\alpha x}{2i\pi x} + O\left(\frac{\operatorname{Log}^2 |x|}{|x|^2}\right). \tag{4.10}$$

To prove the lemma, we transform the equation (4.9) under the linear transformation $\zeta = \log_\alpha x - v$. It then takes the form

$$e^{-v} - 1 - \frac{v}{2i\pi x} + \frac{\log_\alpha x}{2i\pi x} = 0$$

or, with $\sigma = 1/2i\pi x$, $\tau = (\log_\alpha x)/2i\pi x$,

$$e^{-v} - 1 - \sigma v + \tau = 0. \tag{4.11}$$

But this equation is precisely Eq. (2.4.6) in [6], where it is proved that there exist positive numbers a and b such that if $|\sigma| < a$, $|\tau| < a$, Eq. (4.11) has just one solution in the domain $|v| < b$. Furthermore, this solution is the sum of an absolutely convergent double power series

$$v = \tau \sum_{k \geqslant 0} \sum_{m \geqslant 0} c_{km} \sigma^k \tau^m, \qquad |\sigma| < a,\ |\tau| < a,$$

where the c_{km} are constants, $c_{00} = 1$ and $c_{01} = 0$. This proves (4.10). In particular, if $|x|$ is large and $\varepsilon + \alpha < \arg x < 2\pi + \alpha - \varepsilon$ with $\alpha = -\pi$ or zero, then the solution ζ of (4.10) is inside V (Fig. 3.2) or W (Fig. 2.2).

LEMMA 4.4. *Consider the function* $\xi(\omega) = \{2(e^\sigma - \omega - 1)\}^{1/2}$, *where in a neighborhood of* $\omega = 0$ *the sign of the square root has been chosen such that* $\xi(\omega) = \omega + \cdots$. *Then this function maps conformally the strip* $|\mathrm{Im}\,\omega| < 2\pi$ *onto the region S consisting of the entire ξ-plane cut along two hyperbolic arcs described by*

$$\mathrm{Im}\,\xi \cdot \mathrm{Re}\,\xi = \pm 2\pi, \qquad \mathrm{Re}\,\xi \leqslant -|\mathrm{Im}\,\xi|.$$

This lemma is proved in [6, pp. 124–125].

We can now start to study the behavior of the the solutions G_n and F_n. We use saddle point analysis. To explain our method, we begin with the solution F_0 given by the integral (3.17)

$$F_0(x) = \frac{1}{2i\pi} e^{-i\pi/4} \int_V e^{-(i/4\pi)(2i\pi x - w)^2 + e^w}\, dw, \tag{4.12}$$

where V, as in Fig. 3.2, consists of the line segments $(-i\pi + \infty, -i\pi)$, $[-i\pi, i\pi]$ and $(i\pi, i\pi + \infty)$. Consider the function

$$\varphi(w, x) = -\frac{i}{4\pi}(2i\pi x - w)^2 + e^w.$$

Then the saddle points ζ of (3.16) are given by

$$\frac{\partial}{\partial w}\varphi(w, x) = 0 \Leftrightarrow 2i\pi e^{w} + w - 2i\pi x = 0,$$

and, by Lemma 4.3, there is just one saddle point ζ, $\operatorname{Re}\zeta > 0$, $|\operatorname{Im}\zeta| < \pi$, given by (4.10). Under the substitution $w = \zeta = \zeta + \omega$, the integral (3.17) becomes, with $\lambda = 2i\pi e^{\zeta}$,

$$F_0(x) = \frac{1}{2i\pi} e^{-i\pi/4} \exp\left(\frac{\lambda^2 + 2\lambda}{4i\pi}\right) \int_{V-\zeta} \exp\left\{\frac{\omega^2}{4i\pi} + \frac{\lambda}{4i\pi}(e^{\omega} - \omega - 1)\right\} d\omega. \tag{4.13}$$

By Cauchy's theorem, a horizontal shift of the path $V - \zeta$ has no influence on the integral (3.19). Hence,

$$F_0(x) = \frac{1}{2i\pi} e^{-i\pi/4} \exp\left(\frac{\lambda^2 + 2\lambda}{4i\pi}\right) \times \int_{V - i\operatorname{Im}\zeta} \exp\left\{\frac{\omega^2}{4i\pi} + \frac{\lambda}{4i\pi}(e^{\omega} - \omega - 1)\right\} d\omega. \tag{4.13bis}$$

The new path $V - i\operatorname{Im}\zeta$ passes through the saddle point $\omega = 0$. Now we shall follow closely the discussion of [6, Sect. 6.9]. We introduce first the function ξ of Lemma 4.4 as a new integration variable in the integral (4.13bis). We have

$$\xi^2(\omega) = 2(e^{\omega} - \omega - 1), \qquad \frac{d\omega}{d\xi} = \xi \frac{1}{e^{\omega} - 1}.$$

In the strip $|\operatorname{Im}\omega| \leqslant 2\pi$, $e^{\omega} - 1 = 0$ only if $\omega = 0$ or $\omega = \pm 2i\pi$, that is $\xi = 0$ or $\xi^2 = \mp 4i\pi$. Therefore, the function $d\omega/d\xi$ is analytic for ξ in S (cf. Lemma 4.4) and satisfies

$$\frac{d\omega}{d\xi} = O(\xi) \text{ for } |\arg\xi| \leqslant \frac{3}{4}\pi, \qquad |\xi| \geqslant 4.$$

The image C of $V - i\operatorname{Im}\zeta$ under the conformal mapping ξ is a curve starting at $e^{i(\pi/2 - \theta/2)}\infty$ ($\theta = \arg x$) and tending to $e^{-i(\pi/2 + \theta/2)}\infty$, avoiding the hyperbolic arcs (cf. Lemma 4.4). For $|x|$ large, we have for v small

$$\frac{\lambda}{2i\pi}(e^{\omega} - \omega - 1) = \frac{1}{2} e^{\zeta}\xi^2 = \frac{1}{2} x e^{-v}\xi^2,$$

and, if we denote by η the argument of ξ, then $\mathrm{Re}((\lambda/2i\pi)(e^{\omega}-\omega-1))$ is negative if and only if

$$\eta \in \left]-\frac{\pi}{2}-\frac{\theta}{2}-\frac{\pi}{4}, -\frac{\pi}{2}-\frac{\theta}{2}+\frac{\pi}{4}\right[\quad \text{or} \quad \eta \in \left]\frac{\pi}{2}-\frac{\theta}{2}-\frac{\pi}{4}, \frac{\pi}{2}-\frac{\theta}{2}+\frac{\pi}{4}\right[. \tag{4.14}$$

On the rays of argument η given by (4.14), the factor $\exp(\lambda/2i\pi)(e^{\omega}-\omega-1)$ is exponentially small and is maximal at the saddle point $\xi=0$. To avoid the hyperbolic arcs, we have to admit only the values $(1/4)\pi<\eta<(3/4)\pi$. Hence we have the system of inequalities to be satisfied for some small positive δ

$$\begin{aligned}
&\text{(i)} && -\frac{\pi}{4}<\eta-\left(-\frac{\pi}{2}-\frac{\theta}{2}\right)<\frac{\pi}{4},\\
&\text{(ii)} && \delta<\theta<2\pi-\delta,\\
&\text{(iii)} && \frac{\pi}{4}<\eta<\frac{3}{4}\pi.
\end{aligned} \tag{4.15}$$

For each θ given by (ii), we can take $\eta=-(1/4)\theta+\pi/2$. But if we want η to be fixed, we take

$$\begin{aligned}
\eta&=\frac{3}{4}\pi-\frac{\delta}{4} \quad &&\text{if} \quad -\pi+\frac{\delta}{2}<\theta<\frac{\delta}{2},\\
\eta&=\frac{\pi}{4}+\frac{\delta}{4} \quad &&\text{if} \quad -\frac{\delta}{2}<\theta<\pi-\frac{\delta}{2}.
\end{aligned} \tag{4.16}$$

The two rays of argument given by such η are satisfactory from the point of view of the saddle point method. If we write (4.13bis) in the form

$$F_0(x)=\frac{e^{-i\pi/4}}{2i\pi}\exp\left(\frac{\lambda^2+2\lambda}{4i\pi}\right)\int_C e^{\lambda\xi^2/4i\pi}\,e^{\omega^2/4i\pi}\,\frac{d\omega}{d\xi}\,d\xi,$$

we can replace the path C by the straight line through the origin from $-e^{i(\pi/4+\delta/4)}\infty$ to $e^{i(\pi/4+\delta/4)}\infty$, if we restrict $\theta=\arg x$ to the interval $-\delta/2<\theta<\pi-\delta/2$. In a neighborhood of $\omega=0$, we have the power series

$$e^{\omega^2/4i\pi}\frac{d\omega}{d\xi}=a_0+a_1\xi+a_2\xi^2+\cdots, \qquad a_0=1,$$

and we have finally obtained an asymptotic series of the form: For every $M = 1, 2, \ldots$, we have, if $-\delta \leqslant \arg x \leqslant \pi - \delta$ $(\delta > 0)$, $|\lambda| \to \infty$:

$$F_0(x) = \exp\left(\frac{\lambda^2 + 2\lambda}{4i\pi}\right) \sum_{k=0}^{M-1} (-1)^k a_{2k} \times \frac{(2k)!}{k!} (i\pi)^k \lambda^{-k-1/2} + O(|\lambda|^{-M-1/2}). \tag{4.17}$$

The asymptotic expansion (4.17) being uniform for large x in the sector $-\delta \leqslant \arg x \leqslant \pi - \delta$, for some small δ. The same result holds in the sector $-\pi + \delta \leqslant \arg x \leqslant \delta$, hence (4.17) is valid for $|x|$ large and $|\arg x| \leqslant \pi - \delta$.

It remains to solve the asymptotic problem for F_0 in a small sector around the negative real axis, of the form $|\pi - \arg x| \leqslant \delta$, this will use Lemma 4.1. In fact, by (3.16) we have

$$F_0(x) = \frac{e^{i\pi x^2}}{2i\pi} \left\{ e^{-i\pi x} \int_0^{\infty} e^{-t} e^{-(i/4\pi)(\log t)^2} t^{-x-1/2}\, dt - e^{i\pi x} \int_0^{\infty} e^{-t} e^{-(i/4\pi)(\log t)^2} t^{-x-3/2}\, dt \right\}.$$

The two integrals can be analyzed using Lemma 4.1. After some easy computations we find, for $|x| \to \infty$, $|\arg x - \pi| \leqslant \delta$:

$$F_0(x) = \frac{e^{i\pi x^2}}{2i\pi} \Gamma(-x+1)\, e^{-(i/4\pi)(\mathrm{Log}\, t)^2} (-x)^{-1/2} \times \left\{ e^{-i\pi x} \left(1 + O\left(\frac{(\mathrm{Log}(-x))^2}{-x}\right)\right) - \frac{e^{i\pi x}}{-x} \left(1 + O\left(\frac{(\mathrm{Log}(-x))^2}{-x}\right)\right)\right\}. \tag{4.18}$$

The method followed here to obtain (4.18) gives only the order of magnitude of the remainder term and does not furnish more exact information about the size of it. However (4.18), together with (4.17), will be enough for our needs.

To solve the asymptotic problem for the functions G_n, the situation is somewhat different. We will use the saddle point method only for ζ in a small sector around the positive real axis. This method is not quite convenient for sectors of the form $\delta \leqslant \arg \zeta \leqslant 2\pi - \delta$ and, in this case, we find another way to work, based on Eqs. (1.3) and (2.12). Let us suppose $|\arg x| < \pi/2$ and consider the integral representation (2.5). We have exactly the same equation (4.9) for the saddle point, which is given by (4.10), that is, if the principal value of the logarithm is used

$$\zeta = \operatorname{Log} x - \frac{\operatorname{Log} x}{2i\pi x} + O\left(\frac{\operatorname{Log}^2 |x|}{|x|^2}\right).$$

The substitution $w = \zeta + \omega$ transforms (2.5) into

$$G_0(x) = e^{i\pi/4} e^{-(\lambda^2 + 2\lambda)/4i\pi} \int_{-\infty - \zeta}^{\infty - \zeta} e^{-\omega^2/4i\pi - \lambda(e^{\omega} - \omega - 1)/2i\pi} \, d\omega, \tag{4.19}$$

where, as in (4.13), $\lambda = 2i\pi e^{\zeta}$. But when $|\arg x| < \pi/2$, the path of integration in (4.19) can be shifted so that

$$G_0(x) = e^{i\pi/4} e^{-(\lambda^2 + 2\lambda)/4i\pi} \int_{-\infty}^{\infty} e^{-\omega^2/4i\pi - \lambda(e^{\omega} - \omega - 1)/2i\pi} \, d\omega.$$

At this point, we can follow exactly the same steps we did in deriving (4.17) to find that for every $M > 0$

$$G_0(x) = 2i\pi\, e^{-(\lambda^2 + 2\lambda)/4i\pi} \sum_{k=0}^{M-1} (-1)^k b_{2k} \times \frac{(2k)!}{k!} (i\pi)^k \lambda^{-k-1/2} + O(|\lambda|^{-M-1/2}) \tag{4.20}$$

uniformly for $|x| \to \infty$, $|\arg x| \leqslant \pi/2 - \delta$, $\delta > 0$. The coefficients b_k are given by the expansion

$$e^{-\omega^2/4i\pi} \frac{d\omega}{d\xi} = b_0 + b_1 \xi + b_2 \xi^2 + \cdots, \qquad b_0 = 1,$$

and ξ has the same meaning as in Lemma 4.4.

For sectors of the form $\delta < \arg x < 2\pi - \delta$, $\delta > 0$, we use another method. From equations (1.3) and (2.12), we get the system

$$\begin{cases} G_0(x+1) = \dfrac{1}{2i\pi} e^{-2i\pi x} G_0'(x), \\ G_0(x) - G_0(x+1) = g(x+1), \end{cases} \tag{4.21}$$

where $g(\zeta)$ is the right-hand side of (2.12). The pair of equations (4.21) is a first-order linear differential equation whose general solutions are given by

$$G_0(x) = C \exp(e^{2i\pi x}) - 2i\pi \exp(e^{2i\pi x}) \int_{\infty e^{i\pi/4}}^{x} \exp(-e^{2i\pi t})\, e^{2i\pi t} g(t+1)\, dt, \tag{4.22}$$

where the integration is taken along the line of argument $\pi/4$, from infinity to $|x|$, and then along the circle of center origin and radius $|x|$, from $|x|$ to x. It follows from this and from (4.17) that the integral in (4.22) is convergent. To find what the value of C is, we must investigate the asymptotic behavior of the integral in (4.22). This will require the study of the asymptotic behavior of the function (cf. (2.12))

$$g(x) = e^{i\pi/4} \int_W \exp\left[\frac{i}{4\pi}(2i\pi x - w)^2 - e^w\right] dw.$$

The path W is as in Fig. 2.2. The function g is entire and we need only to know the asymptotic expansion in sectors of the form $\delta \leqslant \arg t \leqslant 2\pi - \delta$, for δ positive and small. The saddle point ζ of $\exp[(i/4\pi)(2i\pi x - w)^2 - e^w]$ is exactly the one given by Lemma 4.3 for $\alpha = 0$ or, in other words, the determination of $\log x$ in the formula (4.10) is chosen in such a way that $0 \leqslant \operatorname{Im} \log x < 2\pi$. By the same method as for the solution F_0, we can arrive at the analogue of the formula (4.13bis)

$$g(x) = e^{i\pi/4} e^{-(\lambda^2 + 2\lambda)/4i\pi} \int_{W - i \operatorname{Im} \zeta} \exp\left[-\frac{\omega^2}{4i\pi} - \frac{\lambda}{2i\pi}(e^\omega - \omega - 1)\right] d\omega. \tag{4.23}$$

The same conformal mapping ξ (Lemma 4.4) sends $W - i \operatorname{Im} \zeta$ onto the curve C starting at $e^{-i\theta/2}\infty$ and tending to $e^{i\pi - i\theta/2}\infty$, avoiding the hyperbolic arcs. Now the condition (4.16) should be replaced by

$$\eta = \frac{1}{4}(3\pi - \delta) \qquad \text{if} \quad \frac{1}{2}\delta < \theta < \pi + \frac{\delta}{2}$$

and

$$\eta = \frac{1}{4}(\pi + \delta) \qquad \text{if} \quad \pi - \frac{\delta}{2} < \theta < 2\pi - \frac{\delta}{2}.$$

The final result is that for every $M > 0$

$$g(x) = 2i\pi e^{-(\lambda^2 + 2\lambda)/4i\pi} \times \sum_{k=0}^{M-1} (-1)^k a_{2k} \frac{(2k)!}{k!} (i\pi)^k \lambda^{-k-1/2} + O(|\lambda|^{-M-1/2}), \tag{4.24}$$

where $\lambda = 2i\pi e^\zeta$. This asymptotic expression holds uniformly for large x in the sector $\delta \leqslant \arg x \leqslant 2\pi - \delta$, where δ is positive and small. To obtain the complete asymptotic expansion of G_0 in (4.22), we need to make more explicit λ as a function of x and $\operatorname{Log} x$. From (4.9), we have

$$\lambda e^{\lambda} = 2i\pi\, e^{2i\pi x} \tag{4.25}$$

and, if we define t, σ, τ, and v by

$$t = 2i\pi e^{2i\pi x}, \qquad \sigma = (\operatorname{Log} t)^{-1},$$

$$\tau = (\operatorname{Log}\operatorname{Log} t)(\operatorname{Log} t)^{-1}, \qquad v = \lambda - (\operatorname{Log} t - \operatorname{Log}\operatorname{Log} t),$$

it follows easily that v satisfies the equation

$$e^{-v} - 1 + \tau - \sigma v = 0,$$

which is similar to (4.11). Hence v can be expressed as a power series in the two variables σ and τ, convergent if both variables are sufficiently small in modulus. This computation yields

$$\lambda = \lambda(x) = 2i\pi x - \operatorname{Log} x + \frac{\operatorname{Log} x}{2i\pi x}$$

$$+ \frac{\operatorname{Log}^2 x}{2(2i\pi x)^2} - \frac{\operatorname{Log} x}{(2i\pi x)^2} + O\left(\frac{\operatorname{Log}^3 x}{x^3}\right). \tag{4.26}$$

From (4.26) we obtain

$$\lambda(x+1) = \lambda(x) + 2i\pi - \frac{1}{x} - \frac{\operatorname{Log} x}{2i\pi x^2}$$

$$+ \left(\frac{1}{2i\pi} + \frac{1}{2}\right)\frac{1}{x^2} + O\left(\frac{\operatorname{Log}^3 x}{x^3}\right) \tag{4.27}$$

$$\frac{\lambda^2(x+1) + 2\lambda(x+1)}{4i\pi} = \frac{\lambda^2(x) + 2\lambda(x)}{4i\pi} + i\pi + 2i\pi x$$

$$- \frac{1}{2x} - \operatorname{Log} x + \frac{\operatorname{Log} x}{2i\pi x} + O\left(\frac{\operatorname{Log}^3 x}{x^3}\right) \tag{4.28}$$

$$\frac{\lambda^2(x) + 2\lambda(x)}{4i\pi} = i\pi x^2 + x - x \operatorname{Log} x$$

$$+ \frac{\operatorname{Log}^2 x}{4i\pi} - \frac{\operatorname{Log}^2 x}{2(2i\pi)^2 x} + O\left(\frac{\operatorname{Log}^3 x}{x^3}\right). \tag{4.29}$$

Hence (4.24) becomes for each integer $M > 0$,

$$g(x+1) = 2i\pi \exp\left(-\frac{\lambda^2(x+1) + 2\lambda(x+1)}{4i\pi}\right)$$
$$\times \sum_{k=0}^{M-1} (-1)^k a_{2k} \frac{(2k)!}{k!} (i\pi)^k \lambda(x+1)^{-k-1/2} + O(|\lambda(x)|^{-M-1/2}) \tag{4.30}$$

for large x, $\delta \leqslant \arg x \leqslant 2\pi - \delta$. To apply this expansion, we introduce

$$\text{(i)} \quad A(t) = -e^{2i\pi t} + 2i\pi t - \frac{\lambda^2(t+1) + 2\lambda(t+1)}{4i\pi}$$
$$-\frac{1}{2} \operatorname{Log} \lambda(t+1) + \operatorname{Log} 2i\pi$$

$$\text{(ii)} \quad b_k(t) = (-1)^k a_{2k} \frac{(2k)!}{k!} (i\pi)^k \lambda(t+1)^{-k}, \qquad k = 0, 1, 2, \ldots \tag{4.31}$$

$$\text{(iii)} \quad \mathscr{A}(t) = \exp(-e^{2i\pi t})\, e^{2i\pi t} g(t+1).$$

We have the following asymptotic expansion for $\mathscr{A}(t)$, with respect to the asymptotic scale $\{\exp(A(t)) \cdot b_k(t)\}_k$, in any sector $\delta \leqslant \arg t \leqslant 2\pi - \delta$, $\delta > 0$:

$$\mathscr{A}(t) = e^{A(t)}(b_0(t) + b_1(t) + \cdots).$$

We now state the following lemma.

LEMMA 4.5. *There exists an asymptotic scale $(A_k)_{k \geqslant 0}$ of analytic functions in the sector $0 < \arg x < 2\pi$, $|x| \geqslant R_0 > 0$, such that*

$$\int_i^x \mathscr{A}(t)\, dt = e^{A(x)}(A_0(x) + A_1(x) + \cdots) \tag{4.32}$$

uniformly in any sector $\delta \leqslant \arg x \leqslant 2\pi - \delta$, $\delta > 0$, $|x| \geqslant R > R_0$.

Proof. Formally, the sequence $(A_k)_k$ satisfies the equations

$$A'(x)\{A_0(x) + A_1(x) + \cdots\} + \{A_0'(x) + A_1'(x) + \cdots\}$$
$$= b_0(x) + b_1(x) + \cdots,$$

and it will be sufficient to have

$$A' A_0 = b_0$$
$$A' A_{m+1} + A_m' = b_m, \qquad m \geqslant 0$$

or

$$A_0=\frac{b_0}{A'},$$
$$A_{m+1}=\frac{b_m}{A'}+(-1)^m\left(\frac{1}{A'(x)}\frac{d}{dx}\right)^m\frac{1}{A'(x)}=\frac{b_m(x)}{A'(x)}+(-1)^m B_m(x). \tag{4.33}$$

We then get a sequence $(A_m)_m$ defined on points where $A'(x)\neq 0$ and the proof of the lemma reduces to show that the sequence $(B_m)_m$ is an asymptotic scale. This depends upon the explicit computation of the B_m. For fixed m, B_m is given by the finite sum

$$B_m=A^{-(2m+1)}\sum_{n_0,\ldots,n_m}\alpha_{n_0\cdots n_m}A^{n_0}A^{(1)n_1}\cdots A^{(m)n_m}, \tag{4.34}$$

where the sum is taken over all the $(n_0, \ldots, n_m)$ such that

$$n_0+\cdots+n_{m-1}+n_m=m,$$
$$n_1+\cdots+(m-1)n_{m-1}+mn_m=m,$$

and $A^{(\alpha)\beta}$ stands for $((d^\alpha/dx^\alpha)A)^\beta$. We note that one multi-index is given by $n_2=\cdots=n_m=0$, $n_1=m$, and $n_0=0$. We always have $n_0\leqslant m-1$ with equality if $n_1=\cdots=n_{m-1}=0$, $n_m=1$. Thus $A^{m-1}A^{(m)}$ is the term in the sum B_m where the power of A is maximal and, under some natural conditions on A, it will be the dominating term in it. From (4.26), we obtain the following formula for Log $\lambda(x)$:

$$\operatorname{Log}\lambda=\operatorname{Log}\lambda(x)=\operatorname{Log}x+\operatorname{Log}2i\pi-\frac{\operatorname{Log}x}{2i\pi x}+O\left(\frac{\operatorname{Log}^2 x}{x^2}\right). \tag{4.35}$$

Here the determination of the Log is given by $0\leqslant \operatorname{Im}\operatorname{Log}x<2\pi$. This implies, according to (4.31), (4.28) and (4.29) that

$$A'(t)=-2i\pi e^{2i\pi t}-2i\pi t+\operatorname{Log}t-\frac{\operatorname{Log}t}{2i\pi t}+\frac{1}{2t}+O\left(\frac{\operatorname{Log}^2 t}{t^2}\right). \tag{4.36}$$

It follows that for $m\geqslant 3$

$$A^{(m)}(t)=-(2i\pi)^m e^{2i\pi t}+(-1)^{m-2}\frac{(m-2)!}{t^{m-1}}+O\left(\frac{\operatorname{Log}t}{t^m}\right)$$

and for $m\geqslant 3$

$$B_{m+1}(t)=O(B_m(t)),\qquad |t|\to\infty,\ 0<\arg t<2\pi.$$

Hence (A_m) is an asymptotic scale. ▮

A comparison of the expansions (4.20) and (4.30) shows that the constant C in (4.22) is equal to zero. Therefore, we have obtained the following asymptotic expansion of the function $G_0(x)$ in the sector $0 < \arg x < 2\pi$, $|x| \to \infty$:

$$G_0(x) = -(2i\pi)^2 e^{2i\pi x} \exp\left[-\frac{\lambda^2(x+1) + 2\lambda(x+1)}{4i\pi} - \frac{1}{2} \operatorname{Log} \lambda(x+1) \right]$$
$$\times \{A_0(x) + A_1(x) + \cdots\}. \tag{4.37}$$

Here A_n is given by (4.33), B_n is given by (4.34), and b_n is given by

$$b_n(x) = (-1)^n a_{2n} \frac{(2n)!}{n!} (i\pi)^n \lambda(x+1)^{-n}, \qquad n \geqslant 0.$$

We can easily verify that the asymptotic expansions (4.20) and (4.37) are equivalent in the common sector $0 < |\arg x| < \pi/2$.

5. Biorthogonality and the Series Expansion of Arbitrary Solutions

In this section we prove the main result of this paper. The expansion of arbitrary entire solutions of the equations (1.3) and (3.3) in terms of the solutions G_n and F_n, $n = 0, \pm 1, \pm 2, \ldots$. The proof will partly use the following lemma.

Lemma 5.1. *Let $\lambda = \lambda(x)$ be given by (4.26), then we have*

$$F_0(x)\, G_0(x) = 2i\pi\lambda^{-1}(x) + O(\lambda(x)^{-2}), \qquad x \to \infty, \tag{5.1}$$
$$F_0(x)\, G_0(x) = -2i\pi\lambda^{-1}(x) + O(\lambda(x)^{-2}), \qquad x \to -\infty. \tag{5.2}$$

The first relation also holds for $|\arg x| \leqslant \pi/2 - \delta$, $\delta > 0$ arbitrary.

Proof. The asymptotic behavior (5.1) for $|\arg x| < \pi/2$ follows immediately from (4.17) and (4.20).

On the negative part of the real axis, we rewrite the relation (4.37). Using (4.26), (4.27), (4.28), and (4.29),

$$G_0(x) = -(2i\pi)^2 e^{i\pi x^2 - x + x\log x - \log^2 x/4i\pi + \log x + O(\log^2 x/x)}$$
$$\times \lambda(x+1)^{-1/2} A_0(x) \tag{4.37'}$$

for $|x| \to +\infty$, $\delta < \arg x < 2\pi - \delta$, $\delta > 0$. We rewrite (4.18) in the form

$$F_0(x) = \frac{e^{i\pi x^2}}{2i\pi} \sqrt{2\pi}\, e^{-(x+1/2)\log x + x - (i/4\pi)(\log x)^2 + (i\pi/4) + O(\log^2 x/x)} \tag{4.18'}$$

$|x| \to \infty$, $|\arg x - \pi| < \delta$, $\delta > 0$, and where, in the two relations, the determination of the log function is chosen such that $0 < \operatorname{Im} \log x < 2\pi$. The statement (5.2) easily follows. ∎

We shall also need the following lemma.

LEMMA 5.2. *Let ρ be a fixed complex number or a quantity which is small with respect to x, for instance $\rho = O(x^{1/4})$, then*

$$F_0(x+\rho) = F_0(x) \exp\left\{\rho\left(2i\pi x - \log x + \frac{\log x}{2i\pi x}\right) + \rho^2 \cdot \left(i\pi - \frac{1}{2x}\right) + o\left(\frac{1}{x}\right)\right\}. \tag{5.3}$$

Moreover, for every fixed integer $n = \pm 1, \pm 2, \ldots$, we have for $|x| \to \infty$

$$F_n(x) = F_0(x)(-1)^n \exp\left\{n\left(2i\pi x - \log x + \frac{\log x}{2i\pi x} - \frac{n}{2x}\right) + o\left(\frac{1}{x}\right)\right\} \tag{5.4}$$

and uniformly on compact sets we have for $|n| \to \infty$

$$F_n(x) = F_n(0) \exp\left\{(2i\pi n - \log n)x + i\pi x^2 + O\left(\frac{\operatorname{Log}|n|}{|n|}\right)\right\}. \tag{5.5}$$

Proof. The relation (5.3) is a direct consequence of (4.17) and (4.18) with their respective determination of the logarithm. By taking $\rho = n$ in (5.3), we obtain (5.4) and by fixing $x = \rho$ on compact sets of the plane, and taking $x = n$ in (5.3), we obtain (5.5). ∎

As a consequence of these two lemmas we can evaluate the bilinear product $\{F_n, G_m\}$ defined by (3.5) for any pair of integers n, m. We observe that when $|\arg x| \leqslant \pi/2 - \delta$ $(\delta > 0)$, then

$$F_n(x)\, G_m(x) = (-1)^{n-m} \exp\left\{(n-m)\, 2i\pi x - (n-m+1) \operatorname{Log}(x+m) + O\left(\frac{1}{x}\right)\right\},$$

when $|x| \to \infty$. Noting that $F_n(x)\, G_m(x+1) = F_n(x)\, G_{m+1}(x)$, we have

$$e^{2\pi i x} F_n(x)\, G_m(x+1) = (-1)^{n-m+1} \exp\left\{(n-m)\, 2i\pi x - (n-m) \operatorname{Log}(x+m+1) + O\left(\frac{1}{x}\right)\right\}.$$

Consider first the case $n > m$, then we can evaluate the formula

$$\{F_n, G_m\} = F_n(x)\, G_m(x) - 2i\pi \int_{x-1}^{x} e^{2i\pi t} F_n(t)\, G_m(t+1)\, dt$$

by letting $\arg x = 0$, $x \to \infty$. In this case, both the first term and the integrand are $o(1)$, hence $\{F_n, G_m\} = 0$. In the case $n < m$, we use the same strategy with $\arg x = -\delta$, $0 < \delta < \pi/2$, and again obtain $\{F_n, G_m\} = 0$. Finally, when $x > 0$, $x \to \infty$, we have

$$\{F_n, G_n\} = o(1) + 2i\pi \int_{x-1}^{x} \exp\left\{O\left(\frac{1}{t}\right)\right\} dt \to 2\pi i.$$

We have therefore shown that the following biorthogonality relation holds:

$$\{F_n, G_m\} = 2i\pi\delta_{m,n}. \tag{5.6}$$

On the other hand, if we consider the functions $\tilde{F}$, $\tilde{G}$ given by

$$\tilde{F}(x) = \exp(-e^{2i\pi x}), \qquad \tilde{G}(x) = \exp(e^{2i\pi x})$$

then, as it is easily seen from the definition (3.5)

$$\{\tilde{F}, \tilde{G}\} = 1. \tag{5.7}$$

From the periodicity of the functions $\tilde{F}$, $\tilde{G}$, we get that the products

$$\{\tilde{F}, G_n\},\ \{F_n, \tilde{G}\} \tag{5.8}$$

are independent of n. As it will be seen below, this implies that $\{\tilde{F}, G_n\} = \{F_n, \tilde{G}\} = 0$ for all $z \in \mathbb{Z}$.

LEMMA 5.3. *Let G be an arbitrary entire solution of the equation* (0.1). $G(x+1) = (1/2i\pi)\, e^{-2i\pi x} G'(x)$ *and let C_n be the complex number given by*

$$C_n = \{F_n, G\} = F_n(z)\, G(z) - 2i\pi \int_{z-1}^{z} e^{2i\pi x} F_n(x)\, G(x+1)\, dx.$$

Then for each $\tau > 0$, there is a constant $K(\tau)$, independent of n, such that

$$|C_n| \leqslant K(\tau)\, |F_n(0)|\, e^{-2\pi |n| \tau}. \tag{5.9}$$

Proof. In fact, if we fix τ real, then

$$C_n = F_n(i\tau + 1)\, G(i\tau + 1) - 2i\pi \int_{i\tau}^{i\tau+1} e^{2i\pi x} F_n(x)\, G(x+1)\, dx$$

and then

$$|C_n| \leqslant K_1(\tau) \max_{0 \leqslant u \leqslant 1} |F_n(i\tau + u)|.$$

The estimate (5.5) implies

$$|C_n| \leqslant K_1(\tau)\, |F_n(0)| \times \left| \exp\left\{(2i\pi n - \log n)(i\tau + u) + i\pi(i\tau + u)^2 + O\left(\frac{\log n}{n}\right)\right\}\right|.$$

Hence, for every $\tau > 0$

$$|C_n| \leqslant K(\tau)\, |F_n(0)|\, e^{-2\pi |n| \tau}, \qquad n = 0, \pm 1, \pm 2, \ldots \quad \blacksquare$$

In particular, by (4.17), (4.28) and (4.35), we obtain for every $\tau > 0$ and for every $n > 0$

$$|C_n| \leqslant K(\tau)\, e^{-n \log n + n - 1/2 \log n - 2\pi\tau n} \tag{5.10}$$

and by (4.18), we obtain for every $\tau > 0$ and every $n < 0$

$$|C_n| \leqslant K(\tau)\, e^{-n \log |n| + n - 2\pi\tau |n|}. \tag{5.11}$$

When we apply estimate (5.10) to $\tilde{G}$, we have $|\{F_0, \tilde{G}\}| = |\{F_n, \tilde{G}\}| = o(1)$ as $n \to \infty$. Hence $\{F_n, \tilde{G}\} = 0$ for every $n \in \mathbb{Z}$.

To obtain the same result for $\{\tilde{F}, G_n\}$ we need to observe that the expansions (4.10) and (4.27) imply the following: For any compact subset K of $\mathbb{C}$, there are two constants $a > 0$, $\kappa > 0$ such that

$$|G_n(x)| \leqslant \kappa \exp(n \log |n| + a\,|n|), \qquad x \in K, n \in \mathbb{Z}. \tag{5.12}$$

It is now immediate that $\{\tilde{F}, G_0\} = \{\tilde{F}, G_n\} = o(1)$ when $n \to -\infty$. Hence $\{\tilde{F}, G_n\} = 0$ for every $n \in \mathbb{Z}$.

To investigate the series expansion of an arbitrary solution G of the equation (1.3), we use the integral representation of G in terms of the Green function

$$G(x) = G(x, c)\, G(c) - 2i\pi \int_{c-1}^{c} e^{2i\pi s} G(x, s)\, G(s+1)\, ds.$$

Hence, formally, by Theorem 3.6

$$G(x) = \tilde{C}\tilde{G}(x) - \frac{1}{2i\pi} \sum_{n=-\infty}^{+\infty} C_n G_n(x) \tag{5.13}$$

with

$$\tilde{C} = \{\tilde{F}, G\} = \tilde{F}(c)\, G(c) - 2i\pi \int_{c-1}^{c} e^{2i\pi s}\, \tilde{F}(s)\, G(s+1)\, ds$$

$$C_n = \{F_n, G\} = F_n(c)\, G(c) - 2i\pi \int_{c-1}^{c} e^{2i\pi s}\, F_n(s)\, G(s+1)\, ds.$$

To study the convergence of the series (5.13), we observe (see (5.1), (5.2), and (5.5)) that for x, y real, $x > y$,

$$F_n(x)\, G_n(y) = e^{i\pi(x^2-y^2) + i\pi n(x-y)}\, n^{-1-(x-y)} \left(1 + O\left(\frac{\log |n|}{n}\right)\right),$$

and therefore, the series $\sum F_n(x)\, G_n(y)$ is absolutely convergent with respect to x in any interval $]y, +\infty[$ and uniformly on any compact set of this interval. The same result holds for any complex number y and the interval $\{x \in \mathbb{C},\ x - y > 0\}$. This shows that the relation (5.11) is valid on any compact set of the real line. If we prove that the right-hand side of (5.13) is an entire function, then by means of the analytic continuation, we will have the identity (5.13) valid in the entire complex plane. But estimates (5.10), (5.11), and (5.12) guarantee that the formal series (5.13) represents an entire function.

We can summarize the above results in the following theorem.

THEOREM 5.4. *Let $G(x)$ be an entire solution of the linear difference–differential equation* (1.3), *then*

$$G(x) = \{\tilde{F}, G\}\, \tilde{G}(x) - \frac{1}{2i\pi} \sum_{n=-\infty}^{+\infty} \{F_n, G\}\, G_n(x).$$

The series converges absolutely and uniformly with respect to x on any compact set of the complex plane. Furthermore, the coefficients satisfy the estimates (5.10) *and* (5.11).

Conversely, any series

$$\tilde{c}\tilde{G}(x) - \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} c_n G_n(x)$$

that converges locally uniformly in the plane represents an entire function solution of (1.3), the coefficients satisfy (5.10) and (5.11), and they are uniquely determined.

ACKNOWLEDGMENT

We would like to thank J. Ecalle and R. Gay for many kind suggestions.

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